2006 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
C	D	D	E	E	A	D	C	E	D	A

12	13	14	15	16	17	18	19	20	21	22
C	E	A	B	A	D	D	D	A	E	B

Q1 $y=\frac{2x^2+x-3}{x^2+7x-8}=\frac{(2x+3)(x-1)}{(x+8)(x-1)}=\frac{2x+3}{x+8}$ and $x\neq 1$.

$\therefore y=2-\frac{13}{x+8}$. Two asymptotes: $x=-8$ and $y=2$.

Q2 Solve $y=\frac{x}{2}$ and $y=-\frac{x}{2}-2$ simultaneously to find the two asymptotes intersect at $(-2,-1)$. $\therefore$ the hyperbola is translated 2 left and 1 down. Also $\frac{b}{a}=\pm\frac{1}{2}$, $\therefore \frac{b^2}{a^2}=\frac{1}{4}=\frac{2}{8}$.

Q3 $z^4+1=0$, $(z^2+i)(z^2-i)=0$,

$\therefore z^2=-i=cis\left(-\frac{\pi}{2}+2n\pi\right)$, $\therefore z=cis\left(-\frac{\pi}{4}\right)$ or $cis\left(\frac{3\pi}{4}\right)$,

or $z^2=i=cis\left(\frac{\pi}{2}+2n\pi\right)$, $\therefore z=cis\left(\frac{\pi}{4}\right)$ or $cis\left(-\frac{3\pi}{4}\right)$.

Q4 $z\overline{w}=\left(\sqrt{2}-3i\right)\left(3-i\sqrt{2}\right)=-11i$.

$\therefore \left(z\overline{w}\right)^{-1}=\frac{1}{z\overline{w}}=\frac{1}{-11i}=\frac{i}{11}$.

Q5 $z=-1.82+0.91i$ is in the second quadrant.

$|z|=\sqrt{(-1.82)^2+0.91^2}=2.035$, $\therefore 2<z<4$ and

$Arg(z)=\tan^{-1}\left(\frac{0.91}{-1.82}\right)=2.678\geq\frac{5\pi}{6}$.

Q6 $\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{(\cos x-\sin x)(\cos x-\sin x)}{(\cos x+\sin x)(\cos x-\sin x)}$

$=\frac{\cos^2 x-2\sin x\cos x+\sin^2 x}{\cos^2 x-\sin^2 x}=\frac{1-2\sin x\cos x}{\cos^2 x-\sin^2 x}$

$=\frac{1-\sin(2x)}{\cos(2x)}=\frac{1}{\cos(2x)}-\frac{\sin(2x)}{\cos(2x)}=\sec(2x)-\tan(2x)$.

Q7 $y=3\sec\left(\frac{x-\pi}{2}\right)+1$, $0<x\leq\pi$. The range is $[4,\infty)$.

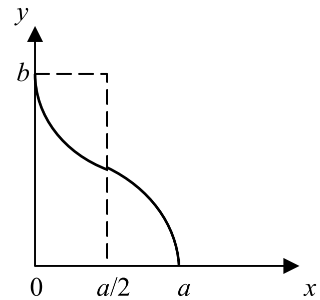
Inverse: Equation is $x=3\sec\left(\frac{y-\pi}{2}\right)+1$, $\therefore \sec\left(\frac{y-\pi}{2}\right)=\frac{x-1}{3}$,

$\therefore \cos\left(\frac{y-\pi}{2}\right)=\frac{3}{x-1}$, $\frac{y-\pi}{2}=\cos^{-1}\left(\frac{3}{x-1}\right)$,

$y=2\cos^{-1}\left(\frac{3}{x-1}\right)+\pi$.

$\therefore f^{-1}(x)=2\cos^{-1}\left(\frac{3}{x-1}\right)+\pi$, domain is $[4,\infty)$.

Q8 $y=\frac{b}{\pi}\cos^{-1}\left(\frac{2x-a}{a}\right)$,



$\int_0^b xdy$ is the area of the region bounded by the curve, the x-axis and the y-axis. This area is exactly equal to the area of the rectangle (dotted) $=\frac{ab}{2}$.

Q9 $x^2-y^2=\frac{3}{4}$. Implicit differentiation, $2x-2y\frac{dy}{dx}=0$,

$\therefore \frac{dy}{dx}=\frac{x}{y}$. At the point where the gradient is 2, $\frac{x}{y}=2$,

$\therefore x=2y$, $(2y)^2-y^2=\frac{3}{4}$, $3y^2=\frac{3}{4}$, $\therefore y=\pm\frac{1}{2}$, $x=\pm 1$.

Hence $\left(-1,-\frac{1}{2}\right)$, $\left(1,\frac{1}{2}\right)$.

Q10 $y=\log_e|x+1|$.

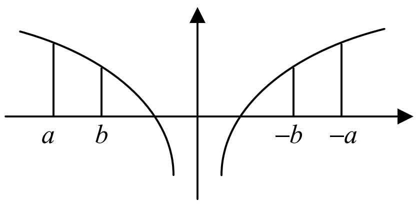
For $x+1>0$, $y=\log_e(x+1)$. When $y=1$, $x+1=e$,

$\frac{dy}{dx}=\frac{1}{x+1}=\frac{1}{e}=e^{-1}$.

For $x+1<0$, $y=\log_e\left(^-(x+1)\right)$. When $y=1$, $^-(x+1)=e$,

$\frac{dy}{dx}=\frac{1}{x+1}=\frac{1}{-e}=-e^{-1}$.

Q11 The graph of $y=\log_e|x|$ is shown below.



$\int_a^b \log_e|x|dx$ is the area of the region on the left and it is equal to the area of the region on the right, i.e. $\int_{-b}^{-a}\log_e|x|dx$. **B** and **C** are the negatives of **A**. **D** and **E** are undefined.

Q12 The shaded region is the first quadrant of the circle of radius 3 cm centred at $(3,0)$. The solid formed by rotation about the x-axis is a hemisphere. $\therefore$ volume $V=\frac{1}{2}\left(\frac{4}{3}\pi 3^3\right)=18\pi$ cm^3.

Q13 The solution to the differential equation could be $y=ax^3+c$, $\therefore \frac{dy}{dx}=3ax^2=kx^2$.

Q14 Note: $\cot\left(\frac{\pi}{2}-x\right)=\tan(x)$.

$$\int_0^{\frac{\pi}{3}}\cot\left(\frac{\pi}{2}-x\right)dx=\int_0^{\frac{\pi}{3}}\tan(x)dx=\int_0^{\frac{\pi}{3}}\frac{\sin(x)}{\cos(x)}dx$$

$$=\int_0^{\frac{\pi}{3}}\left(-\frac{1}{u}\frac{du}{dx}\right)dx=\int_1^{\frac{1}{2}}\left(-\frac{1}{u}\right)du$$

$$=\int_{\frac{1}{2}}^1\frac{1}{u}du=\int_{\frac{1}{2}}^1\frac{du}{u} \text{ or } \int_{\frac{1}{2}}^1\frac{dx}{x}.$$

Let $u=\cos x$,
$-\frac{du}{dx}=\sin x$.
$x=0$, $u=1$.
$x=\frac{\pi}{3}$, $u=\frac{1}{2}$.

Q15 $\frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$.

$x=1$, $y=-2$.

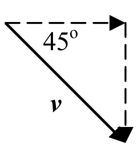
$x=1.2$, $y\approx -2+\frac{0.2}{\sqrt{1+1^2}}=-2+\frac{0.2}{\sqrt{2}}$.

$x=1.4$, $y\approx -2+\frac{0.2}{\sqrt{2}}+\frac{0.2}{\sqrt{1+1.2^2}}=-2+\frac{0.2}{\sqrt{2}}+\frac{0.2}{\sqrt{2.44}}$.

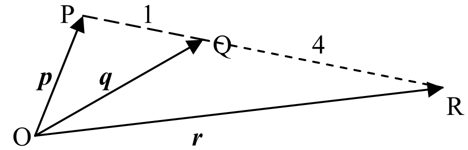
Q16 Direction of motion is given by the direction of velocity vector.

$\mathbf{r}(t)=3\sin(t)\mathbf{i}+\sqrt{3}\cos(t)\mathbf{j}$, $\mathbf{v}(t)=3\cos(t)\mathbf{i}-\sqrt{3}\sin(t)\mathbf{j}$.

At $t=\frac{\pi}{3}$, $\mathbf{v}(t)=3\cos\left(\frac{\pi}{3}\right)\mathbf{i}-\sqrt{3}\sin\left(\frac{\pi}{3}\right)\mathbf{j}=\frac{3}{2}\mathbf{i}-\frac{3}{2}\mathbf{j}$.



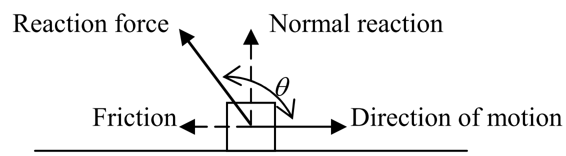
Q17



$\overrightarrow{OQ}=\overrightarrow{OP}+\frac{1}{1+4}\overrightarrow{PR}$,

$\therefore \mathbf{q}=\mathbf{p}+\frac{1}{5}(\mathbf{r}-\mathbf{p})=\mathbf{p}+\frac{1}{5}\mathbf{r}-\frac{1}{5}\mathbf{p}=\frac{4}{5}\mathbf{p}+\frac{1}{5}\mathbf{r}=\frac{1}{5}(4\mathbf{p}+\mathbf{r})$.

Q18 The reaction force on the particle consists of two components: the normal reaction force and the force of friction, both are exerted by the surface on the particle.



Q19 Momentum $\mathbf{p}=m\mathbf{v}(t)=2(\cos(2t)\mathbf{i}-5\mathbf{j})$.

$\frac{d}{dt}\mathbf{p}=-4\sin(2t)\mathbf{i}$. At $t=\frac{\pi}{4}$, $\frac{d}{dt}\mathbf{p}=-4\mathbf{i}$.

$\therefore$ the magnitude of $\frac{d}{dt}\mathbf{p}=4$.

Q20 $u=0$, $a=-9.8$.

At $t=2$, $s=ut+\frac{1}{2}at^2=-19.6$.

At $t=3$, $s=ut+\frac{1}{2}at^2=-44.1$.

Displacement in the third second $-44.1-(-19.6)=-24.5$.

Distance $=24.5$ m.

Q21 For $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to be linearly dependent, there are non-zero $m,n\in R$ such that $\mathbf{a}=m\mathbf{b}+n\mathbf{c}$, i.e.

$3\mathbf{i}+p\mathbf{j}=m(2\mathbf{i}-5\mathbf{j})+n(5\mathbf{i}+2\mathbf{j})$,

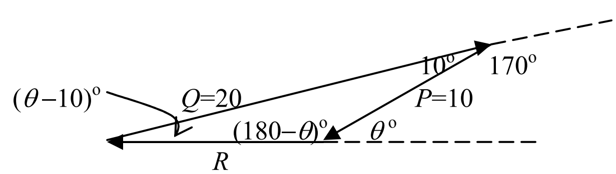
$\therefore 2m+5n=3$ …………(1)

and $-5m+2n=p$……….(2)

$5\times(1)+2\times(2)$, $29n=15+2p$, $n=\frac{15+2p}{29}\neq 0$, $\therefore p\neq -\frac{15}{2}$.

$2\times(1)-5\times(2)$, $29m=6-5p$, $m=\frac{6-5p}{29}\neq 0$, $\therefore p\neq \frac{6}{5}$.

Q22



Use the cosine rule to find R:

$R=\sqrt{20^2+10^2-2(20)(10)\cos 10^o}=10.3$.

Use the sine rule to find θ:

$\frac{\sin(180-\theta)}{20}=\frac{\sin(10)}{10.3}$, $\frac{\sin(\theta)}{20}=\frac{\sin(10)}{10.3}$, $\theta\approx 19.7$ °.

Section 2

Q1a.

θ	0	1	2	3	4	5	6
r	0	1	2	3	4	5	6

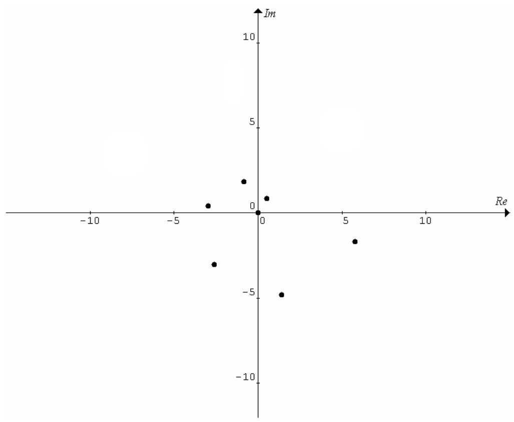
Q1b. $z = rcis\theta = \frac{\pi}{3}cis\left(\frac{\pi}{3}\right) = \frac{\pi}{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= \frac{\pi}{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{\pi}{6} + \frac{\pi\sqrt{3}}{6}i$.

Q1c. $|w| = \frac{\pi}{2}$. w lies on the Im-axis and $0 < \arg w < \pi$,

$\therefore \arg z = \frac{\pi}{2}$. $\therefore |w| = \arg w$. Hence $w \in S$.

Q1d.



Q1e. If $rcis\theta \in S$, then its conjugate is $rcis(-\theta) \in T$.

$\therefore T = \{z : |z| = -\arg z\}$ where $\arg z \in (-\infty, 0]$.

Q2a. $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \left(e^{t-0.5} - \log_e(t+0.5)\right)\boldsymbol{i} - \boldsymbol{j},\ 0 \le t \le 1$.

$\left|\overrightarrow{PQ}\right| = \sqrt{\left(e^{t-0.5} - \log_e(t+0.5)\right)^2 + (-1)^2} = \sqrt{\left(e^{t-0.5} - \log_e(t+0.5)\right)^2 + 1}$

Q2bi. Use graphics calculator to sketch

$\left|\overrightarrow{PQ}\right| = \sqrt{\left(e^{t-0.5} - \log_e(t+0.5)\right)^2 + 1}$.

The minimum point is $(0.5, 1.414)$. The closest approach is 1.414 and it occurs at $t = 0.5$.

Q2bii. For $0 \le t \le 1$, from the sketch the greatest distance is 1.64 and it occurs at $t = 0$.

Q2c. The two particles move in the same direction when their velocity vectors are parallel,

i.e. $\frac{d}{dt}\boldsymbol{p} = k\frac{d}{dt}\boldsymbol{q}$ where k is a constant.

$\therefore \frac{1}{t+0.5}\boldsymbol{i} + \mathbf{j} = k\left(e^{t-0.5}\right)\boldsymbol{i} + k\boldsymbol{j}$,

$\therefore k = 1$ and $\frac{1}{t+0.5} = e^{t-0.5}$, i.e. $t = 0.5$.

Q2d. For P: $x = \log_e(t+0.5)$, $y = t + 0.5$, $\therefore x = \log_e y$,

$y = e^x$, $0 \le t \le 1$.

For Q: $x = e^{t-0.5}$, $y = t - 0.5$, $\therefore x = e^y$, $y = \log_e x$, $0 \le t \le 1$.

Q2e. For P at $t = 0$, $x = \log_e 0.5$, $y = 0.5$;

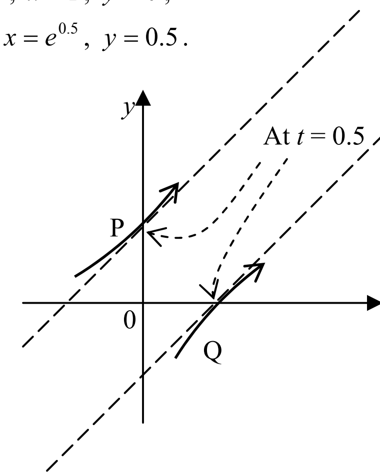
at $t = 0.5$, $x = 0$, $y = 1$;

at $t = 1$, $x = \log_e 1.5$, $y = 1.5$.

For Q at $t = 0$, $x = e^{-0.5}$, $y = -0.5$;

at $t = 0.5$, $x = 1$, $y = 0$;

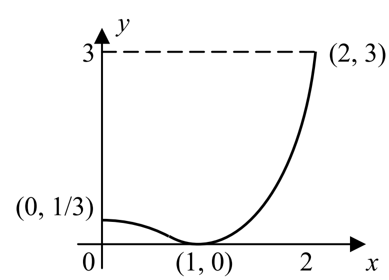
at $t = 1$, $x = e^{0.5}$, $y = 0.5$.



Before $t = 0.5$, P and Q move closer together. At $t = 0.5$, they move parallel to each other (i.e. in the same direction) and are the closest. They move away from each other after $t = 0.5$.

Q3a. The range is $[0,3]$.

Q3b.



Q3c. $y = \frac{1}{3}(x-1)^2(x+1)^2 = \frac{1}{3}(x^2-1)^2 = \frac{1}{3}(x^4 - 2x^2 + 1)$.

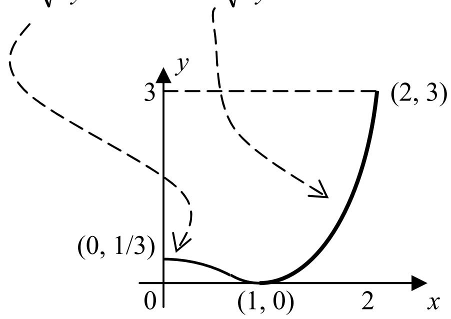
Required area $= \int_0^2 \left[3 - \frac{1}{3}(x^4 - 2x^2 + 1)\right]dx$

$= \left[3x - \frac{1}{3}\left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right)\right]_0^2$

$= \frac{224}{45}$ m^2.

Q3d. $y=\frac{1}{3}(x-1)^2(x+1)^2=\frac{1}{3}(x^2-1)^2$, $\therefore (x^2-1)^2=3y$,

$\therefore x^2=1-\sqrt{3y}$ or $x^2=1+\sqrt{3y}$



$$V=\int_0^3 \pi\left(1+\sqrt{3y}\right)dy-\int_0^{\frac{1}{3}}\pi\left(1-\sqrt{3y}\right)dy$$

$$=\left[\pi\left(y+\frac{2\sqrt{3}y^{\frac{3}{2}}}{3}\right)\right]_0^3-\left[\pi\left(y-\frac{2\sqrt{3}y^{\frac{3}{2}}}{3}\right)\right]_0^{\frac{1}{3}}$$

$$=\frac{80\pi}{9}\text{ m}^3.$$

Q4a. $\overrightarrow{AC}=\boldsymbol{c}-\boldsymbol{a}$, $\overrightarrow{BC}=\boldsymbol{c}-\boldsymbol{b}$ and $\overrightarrow{BA}=\boldsymbol{a}-\boldsymbol{b}$.

Q4b. $\overrightarrow{OM}=\frac{1}{2}(\boldsymbol{c}+\boldsymbol{b})$, $\overrightarrow{ON}=\frac{1}{2}(\boldsymbol{c}+\boldsymbol{a})$ and $\overrightarrow{OP}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

Q4ci. Since $\overline{OM}$ and $\overline{ON}$ are perpendicular to $\overline{BC}$ and $\overline{AC}$ respectively, $\therefore \overrightarrow{OM}\bullet\overrightarrow{BC}=\frac{1}{2}(\boldsymbol{c}+\boldsymbol{b})\bullet(\boldsymbol{c}-\boldsymbol{b})=0$ and

$\overrightarrow{ON}\bullet\overrightarrow{AC}=\frac{1}{2}(\boldsymbol{c}+\boldsymbol{a})\bullet(\boldsymbol{c}-\boldsymbol{a})=0$.

Hence $|\boldsymbol{c}|^2-|\boldsymbol{b}|^2=0$ and $|\boldsymbol{c}|^2-|\boldsymbol{a}|^2=0$, $\therefore |\boldsymbol{a}|=|\boldsymbol{b}|=|\boldsymbol{c}|$.

Q4cii. $\overrightarrow{OP}\bullet\overrightarrow{BA}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}-\boldsymbol{b})=\frac{1}{2}\left[\,|\boldsymbol{a}|^2-|\boldsymbol{b}|^2\,\right]=0$,

$\therefore \overline{OP}$ is perpendicular to $\overline{BA}$.

Q4d. Since $\overrightarrow{AC}=\boldsymbol{c}-\boldsymbol{a}$, $\therefore \overrightarrow{AC}\bullet\overrightarrow{AC}=(\boldsymbol{c}-\boldsymbol{a})\bullet(\boldsymbol{c}-\boldsymbol{a})$,

$\left|\overrightarrow{AC}\right|^2=|\boldsymbol{c}|^2+|\boldsymbol{a}|^2-2|\boldsymbol{c}||\boldsymbol{a}|\cos\alpha=2d^2(1-\cos\alpha)$.

Similarly, $\left|\overrightarrow{BC}\right|^2=|\boldsymbol{c}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{c}||\boldsymbol{b}|\cos\beta=2d^2(1-\cos\beta)$ and

$\left|\overrightarrow{BA}\right|^2=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{a}||\boldsymbol{b}|\cos\gamma=2d^2(1-\cos\gamma)$.

Hence $\left|\overrightarrow{AC}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{BA}\right|^2=2d^2\left[3-(\cos\alpha+\cos\beta+\cos\gamma)\right]$.

Q5a. The particle is slowing down,

$\therefore$ the resultant force $R=-\frac{500}{25-t^2}$.

Newton's second law: $a=\frac{R}{m}$, $\therefore \frac{dv}{dt}=-\frac{100}{25-t^2}$.

Q5b. The particle has an initial velocity 10 ms^{-1}.

At time t the change in velocity $\Delta v=\int_0^t\left(-\frac{100}{25-t^2}\right)dt$

$=\int_0^t\left[-10\left(\frac{1}{5+t}+\frac{1}{5-t}\right)\right]dt$ (Partial fractions)

$=-10\left[\log_e|5+t|-\log_e|5-t|\right]_0^t$

$=-10\log_e\left|\frac{5+t}{5-t}\right|$.

$\therefore$ at time t the velocity $=10+\Delta v=10-10\log_e\left|\frac{5+t}{5-t}\right|$

$=10\left(1-\log_e\left|\frac{5+t}{5-t}\right|\right)$ ms^{-1}.

Q5c. Comes to a stop, $v=0$,

$\therefore 10\left(1-\log_e\left|\frac{5+t}{5-t}\right|\right)=0$, $\therefore \log_e\left|\frac{5+t}{5-t}\right|=1$, $\therefore \left|\frac{5+t}{5-t}\right|=e$.

There are two possible solutions for the last equation:

$\frac{5+t}{5-t}=e$ or $\frac{5+t}{5-t}=-e$,

$\therefore t=\frac{5(e-1)}{e+1}$ or $t=\frac{5(e+1)}{e-1}$.

The first solution is correct because it is the earliest time the particle comes to a stop and no further motion after that time.

Q5di. $t=\frac{5(e-1)}{e+1}\approx 2.31$

Stopping distance = magnitude of displacement

$=\int_0^{2.31}10\left(1-\log_e\left(\frac{5+t}{5-t}\right)\right)dt$

Q5dii. Use graphics calculator to sketch

$v=10\left(1-\log_e\left(\frac{5+t}{5-t}\right)\right)$, then evaluate the definite integral.

$\int_0^{2.31}10\left(1-\log_e\left(\frac{5+t}{5-t}\right)\right)dt=12$ m.

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